Section 4.6

Equations with Variables in the Denominator

Fractions have a top part called the numerator and a bottom part called the denominator. All fractions are of the form

\[ \frac{p}{q} \]

where \( p \) and \( q \) any numbers you like, EXCEPT \( q \) cannot be ZERO

Some examples

\[ \frac{2}{3} = \text{good fraction} \]
\[ \frac{0}{4} = 0 \quad \text{good fraction} \]
\[ \frac{4}{0} = \text{undefined fraction, BAD} \]
\[ \frac{0}{0} = \text{indeterminate form, means more work (Calculus I-II)} \]

Example

\[ 3 + \frac{6}{t-3} = \frac{6}{t(t-3)} \]

Notice that some of the denominators have variables in them.

\[ \frac{t-3}{t(t-3)} \]

and you have to make sure that

\[ t - 3 \neq 0 \]
\[ t(t - 3) \neq 0 \]

This means that we do NOT want \( t = 3 \) or \( t = 0 \) since these answers would be BAD in the sense that they would make the problem undefined.

Anytime your problem has denominators in it you must stop and take note of what could make it undefined. That way if your answer turns out to be one of the BAD answers you will know to toss it.

Basic Strategy

1. Factor all denominators. In this section you do not need to worry about this since in this section things are already factored.

2. Find out what numbers would make your problem undefined.
3. Multiply both sides of your equation by SOMETHING that will cause each and every denominator to cancel. This something is the collection of symbols and terms that would be the common denominator. Another way to look at it is to grab each of the unique terms from all of the denominators.

4. After step 3 you should be left with NO denominators and a linear or quadratic equation.

5. Solve the easier equation and then check if any of your answers contain the bad answers from step 2. If they do toss out those numbers. If you toss out all of your answers then your problem has no solution.

Example

\[ 3 + \frac{6}{t-3} = \frac{6}{t^2 - 3t} \]

Note the bad answers are \( t = 0, 3 \). Now multiply by \( t(t - 3) \) to get

\[ t(t - 3) \left[ 3 + \frac{6}{t-3} \right] = t(t - 3) \left[ \frac{6}{t(t-3)} \right] \]

\[ 3t(t - 3) + 6t = 6 \]
\[ 3t^2 - 9t + 6t = 6 \]
\[ 3t^2 - 3t - 6 = 0 \]
\[ t^2 - t - 2 = 0 \]
\[ (t + 1)(t - 2) = 0 \]

\[ t = -1, 2 \]

Neither answer is BAD so we keep them both.

Example

\[ \frac{x}{x-6} - 3 = \frac{6}{x-6} \]

Note the bad answer is 6 so multiply all terms by \( x - 6 \) to get
\[
x - 3(x - 6) = 6 \\
x - 3x + 18 = 6 \\
-2x = -12 \\
x = 6
\]

But \(x = 6\) is bad so we toss it out. Therefore we have no solutions or said a different way the solution set is empty.

**More Word Problems**

**Example**

The ratio of the weight of sodium to chloride in common table salt is 5 to 3. Find the amount of each element in a salt compound weighing 200 pounds.

**Solution**

\[
\frac{5}{3} = \frac{\text{weight of sodium}}{\text{weight of chloride}} = \frac{s}{c}
\]

We know

\[
s + c = 200 \\
s = 200 - c
\]

so

\[
\frac{5}{3} = \frac{\text{weight of sodium}}{\text{weight of chloride}} = \frac{s}{c} \\
\frac{5}{3} = \frac{200 - c}{c}
\]

Cross multiply to get

\[
5c = 3(200 - c) \\
5c = 600 - 3c \\
8c = 600 \\
c = 75 \\
s = 200 - c \\
s = 200 - 75 \\
s = 125
\]

\[
\frac{5}{3} = \frac{\text{weight of sodium}}{\text{weight of chloride}} = \frac{125}{75}
\]
Example

A 20 ft. board is to be cut into 2 pieces whose length as in the ratio of 7 to 3. Find the lengths of the pieces.

First note that this problem is not really "real". If you were really cutting a board into parts you would have to account for the width of the saw blade. In this problem we assume the blade has no width.

We know there are 2 parts so call them part 1 and part 2 and that if we add them up that

\[
\text{part}_1 + \text{part}_2 = 20
\]

\[
\text{part}_1 + \text{part}_2 = 20
\]

We know that

\[
\frac{\text{part}_1}{\text{part}_2} = \frac{7}{3} = \frac{p_1}{p_2}
\]

\[
7p_2 = 3p_1
\]

\[
p_2 = \frac{3p_1}{7}
\]

So

\[
p_1 + p_2 = 20
\]

\[
p_1 + \frac{3p_1}{7} = 20
\]

\[
\frac{7}{7}p_1 + \frac{3}{7}p_1 = 20
\]

\[
\frac{10}{7}p_1 = 20
\]

\[
10p_1 = 140
\]

\[
p_1 = 14\text{ft}
\]

\[
p_2 = 6\text{ft}
\]