Section 8.1

Functions

This section introduces us to functions. Functions are the foundations of Mathematics and will appear very often throughout this course and later courses.

A function simply takes an INPUT and produces an OUTPUT.

The only restriction is that each input can produce only 1 output.

Example

\[ f(x) = 3x + 2 \]

f is the name of the function
( ) is the place or spot where the input is to be placed.
x is the symbol that stands for the word INPUT HERE
3x + 2 is a rule or set of instructions on how to transform input into output.

When I see

\[ f(x) = 3x + 2 \]

what I really see is

\[ f(\text{input goes here}) = 3 \times (\text{same input goes here}) + 2 \]

so

\[ f(2) = 3(2) + 2 = 8 \]
\[ f(B) = 3(B) + 2 = 3B + 2 \]
\[ f(2x - 6) = 3(2x - 6) + 2 = 6x - 18 + 2 = 6x - 16 \]

Make sure that you think about the difference between these 2 functions

\[ f(x) = 5x \]
\[ f(\text{ input here }) = 5 \times (\text{ input here }) \]
\[ g(x) = 5 \]
\[ g(\text{ input here }) = 5 \]
The $f$ function has a place to put the input on the right hand side since there is an $x$ there, but the $g$ function has no place for an $x$ so it only sends out 5.

\[ f(3) = 5 \times 3 = 15 \]
\[ g(3) = 5 \]
\[ f(-1) = 5 \times -1 = -5 \]
\[ g(-1) = 5 \]

**Ordered Pairs**

Sometimes you will see a function as a set of ordered pairs

\[ \{(1, 3), (4, 5), (6, 7)\} \]

The first number that you see is the input and the second number is the output, so $(1, 3)$ means $f(1) = 3$

**Piecewise Defined Functions**

Some functions have more than one rule and they change how they behave depending on where the input comes from.

For example:

\[ f(x) = \begin{cases} 
5 & \text{if } x \leq 1 \\
-x + 3 & \text{if } x > 1 
\end{cases} \]

has 2 different rules.

**Rule 1 or How 1**

If the input $x \leq 1$ then use the rule $f(x) = 5$

**Rule 2 or How 2**

If the input $x > 1$ then use the rule $f(x) = -x + 3$

So

\[ f(2) = -2 + 3 = 1 \text{ since the input } 2 > 1 \]

But

\[ f(-2) = 5 \text{ since the input } -2 \leq 1 \]
Difference Quotients

The text will give us a function \( f(x) \) and ask us to calculate something ugly like:
\[
\frac{f(a + h) - f(a)}{h}
\]

Here is how to do this:

1. Write your function as \( f(_) \) for example if \( f(x) = 3x + 2 \) then this would be
\[
f(_) = 3(\_) + 2
\]

2. Calculate \( f \) of the input \( a + h \) or
\[
f(a + h) = 3(a + h) + 2
= 3a + 3h + 2
\]

3. Calculate \( f \) of the input \( a \) or
\[
f(a) = 3a + 2
\]

4. Simplify the answers to 2 and 3 and find
\[
f(a + h) - f(a) = (3a + 3h + 2) - (3a + 2)
= 3a + 3h + 2 - 3a - 2
= 3h
\]

5. Take the answer to 4 and divide by \( h \) to find
\[
\frac{f(a + h) - f(a)}{h} = \frac{3h}{h} = 3
\]

The reason that we want to calculate this thing is not clear right now but it will be in either Business Calculus or Engineering Calculus.

Preview

If we have two points
\[(a, f(a)) \text{ and } (a + h, f(a + h))\]
and then try to find the slope \( m \) you will see that
\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}
\]

Function or Not

If someone hands you the graph of something how do you know if it is a function or
not?

We use what is called the **Vertical Line Test**. Simply stated just draw vertical lines on the graph. If any of these lines cross the graph in more than 1 place then it is not a function.

This method works because it is showing you places were 1 input produces more than 1 output.

**Domain and Range**

The **DOMAIN** is the collection or set of all valid inputs into the function. You might say how do I know if the input is valid or not.

For our purposes a valid input produces a REAL number as an output. We do not want undefined to come out or a complex number to come out.

The **RANGE** is the collection of numbers that comes out if we dump all of the domain into the function.

**Finding the Domain**

1. Look at your function and see if it has either variables in a denominator or variables under and even rooted radical.

2. If it does not and your problem only has x's in it then the domain is the collection of all Real Numbers or \((-\infty, \infty)\)

3. If you problem has variables in the denominator then you must perform the following steps:
   - Set the denominator = 0 and solve.
   - Remove the solutions for part 1 from \((-\infty, \infty)\) This means put open dots there.
   - What's left on the number line is the domain.

4. If your problem has variables under an even rooted Radical then you must perform the following steps:
   - Grab the stuff from under the radical and solve the **INEQUALITY**
     \[ \text{stuff} \geq 0 \]
   - The solution to this inequality is the domain.
In the discussion of the domain we kept mentioning the even rooted radicals.

You might ask what occurs if it is an odd rooted radical. Usually the domain is \((-\infty, \infty)\)

The reason is this for cube roots:

\[
\sqrt[3]{8} = 2 \\
\sqrt[3]{0} = 0 \\
\sqrt[3]{-8} = -2
\]

We see that putting a negative in a cube root does not produce a complex number output.

**Finding the Range**

There is no shortcut in finding the range.

You must:

1. Make a table of inputs and outputs.

2. Pick a few inputs from the DOMAIN or collection of valid inputs and observe what numbers come out.

3. In your observation of the outputs ask yourself what would happen to the highest and lowest outputs as you picked more and more inputs.

4. Would they stay the same get smaller, get larger or what? If you need to pick more inputs then do that.

5. Make a prediction about the collection of all outputs and this is the range.

**Example**

\[ f(x) = |x| \]

First the domain is \((-\infty, \infty)\) since there is no variables under an even rooted radical or variables in a denominator.

Make a chart
Looking at the outputs 0 was the smallest and 10 was the largest. Now ask what occurs if I pick more inputs, does the smallest, largest stay unchanged or not.

Conclude that the range is \([0, \infty)\)

If you can graph or plot the function on a computer or calculator you can quickly get a visual feel for how high or how low a function is going to go.

From the graph

we see the function goes upward from the x axis forever. The lowest is 0 so the range is \([0, \infty)\)