Section 2.8

Subspaces

Suppose we have a set $S$ which has a collection of things in it

$$S = \{ \ldots \}$$

Also suppose that this set has some "operations" on it.

In our case $S = \mathbb{R}^n$ and the "operations" are addition and scalar multiplication.

If we take a subset of items from $S$ and call this new set $R$ then there are some questions that we can ask:

1. Is the identity element for our operations a member of $R$? For our case is 0 in $R$?
2. If we take two items from $R$, namely $u$ and $v$ and then we add them together to get something new called $u + v$, then is the new thing also in $R$?
3. If we take an item $u$ from $R$ and a scalar number $c$ and then form all possible $cu$ is the new also in $R$?

If 1, 2 and 3 are all true then we call $R$ a sub(insert name). You will also here that 2 and 3 make the operation "closed".

If $S$ is a group then $R$ would be a subgroup
If $S$ is a field then $R$ would be a subfield
If $S$ is a vector space then $R$ is a subspace

These things might seem a little odd but consider your computer screen a subspace. Hopefully if you do some operation you hope the new text or object is on the screen and 100’s of pixels off the screen. We will also revisit theses concepts in section 4.1

Some Facts

Take matrix $A$

$$A = \begin{bmatrix}
1 & 0 & 3 \\
3 & 2 & 1 \\
1 & 4 & 5
\end{bmatrix}$$

and form the following for all possible real numbers for $c_1, c_2$ and $c_3$
\[ v_i = c_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \]

**Fact 1** -> This collection of vectors is called the Column Space for A or Col A.

By construction we see that the Col A is a subspace of \( \mathbb{R}^3 \) in our case. Is it all of \( \mathbb{R}^3 \) or said a different way -> do the column vectors of A span all of \( \mathbb{R}^3 \)?

**Fact 2** -> The collection of vectors that solve the following are called the Null Space of matrix A or Nul A

\[ Ax = 0 \]

**Example 6, pg 151**

Let

\[
\begin{align*}
\begin{bmatrix} 1 \\ -3 \\ 2 \\ 3 \end{bmatrix},& v_2 = \begin{bmatrix} 4 \\ -4 \\ 5 \\ 7 \end{bmatrix},& v_3 = \begin{bmatrix} 5 \\ -3 \\ 6 \\ 5 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} -1 \\ -7 \\ -1 \\ 2 \end{bmatrix}
\end{align*}
\]

Determine if vector u is in the subspace generated by the other three vectors namely can we find scalars so that

\[
u = c_1 v_1 + c_2 v_2 + c_3 v_3
\]

\[
\begin{bmatrix} -1 \\ -7 \\ -1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -4 \\ 5 \\ 7 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ -3 \\ 6 \\ 5 \end{bmatrix}
\]

This means the following should have a consistent solution

\[
\begin{bmatrix} 1 & 4 & 5 & -1 \\ -3 & -4 & -3 & -7 \\ 2 & 5 & 6 & -1 \\ 3 & 7 & 5 & 2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & u \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -6 \end{bmatrix}
\]

which is not consistent.
Notice that the vector

\[
\begin{bmatrix}
9 \\
-7 \\
11 \\
12
\end{bmatrix}
\]

is in the span

\[
u = c_1v_1 + c_2v_2 + c_3v_3
\]

\[
\begin{bmatrix}
9 \\
-7 \\
11 \\
12
\end{bmatrix} = c_1\begin{bmatrix}1 \\ -3 \\ 2 \\ 3\end{bmatrix} + c_2\begin{bmatrix}-4 \\ 5 \\ 7 \\ 3\end{bmatrix} + c_3\begin{bmatrix}5 \\ -3 \\ 6 \\ 5\end{bmatrix}
\]

\[
\begin{bmatrix}v_1 & v_2 & v_3 & u\end{bmatrix}
\Rightarrow
\begin{bmatrix}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix}
\]

\[
u = 0v_1 + v_2 + v_3
\]

\[
\begin{bmatrix}
9 \\
-7 \\
11 \\
12
\end{bmatrix} = \begin{bmatrix}4 \\ -4 \\ 5 \\ 7\end{bmatrix} + \begin{bmatrix}5 \\ -3 \\ 6 \\ 5\end{bmatrix}
\]

**Example 8, pg 151**

\[
v_1 = \begin{bmatrix}-2 \\ 0 \\ 6\end{bmatrix}, v_2 = \begin{bmatrix}-2 \\ 3 \\ 3\end{bmatrix}, v_3 = \begin{bmatrix}0 \\ -5 \\ 5\end{bmatrix} \text{ and } p = \begin{bmatrix}-6 \\ 0 \\ 0 \\ 1\end{bmatrix}
\]

Determine if \(p\) is on Col A.
\[
\begin{align*}
c_1 v_1 + c_2 v_2 + c_3 v_3 &= p \\
c_1 \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -5 \\ 5 \end{bmatrix} &= \begin{bmatrix} -6 \\ 1 \\ 17 \end{bmatrix}
\end{align*}
\]

\[
\begin{bmatrix} v_1 & v_2 & v_3 & p \end{bmatrix}
\begin{bmatrix} -2 & -2 & 0 & -6 \\ 0 & 3 & -5 & 1 \\ 6 & 3 & 5 & 17 \end{bmatrix}
\Rightarrow
\begin{bmatrix} -2 & 0 & -2 & -6 \\ 0 & 3 & -5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\Rightarrow
\begin{bmatrix} 1 & 0 & \frac{5}{3} & \frac{8}{3} \\ 0 & 1 & -\frac{5}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

Example 10, pg 151

Same matrix A as in example 8. Is the following vector u in Nul A

\[
A = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 3 & -5 \\ 6 & 3 & 5 \end{bmatrix}, \quad u = \begin{bmatrix} -5 \\ 5 \\ 3 \end{bmatrix}
\]

This is easy to check since if u is in Nul A then

\[
Au = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 3 & -5 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} -5 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

and we conclude that u is in Nul A.

Example 16, 18, pg 151
Determine if the sets form a basis in $\mathbb{R}^2$ and $\mathbb{R}^3$

$v_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ -10 \end{bmatrix}$

If this is a basis then they are linearly independent and the only solution to $Av = 0$ is the 0 vector

$\begin{bmatrix} -2 & 4 & 0 \\ 5 & -10 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

but in our case we have multiple solutions which means that they are not linearly independent. Also if you look closely $v_2 = -2v_1$

$v_1 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}$

If this is a basis then they are linearly independent and the only solution to $Av = 0$ is the 0 vector

$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 1 & -1 & 1 & 0 \\ -3 & 2 & -4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

but in our case we have multiple solutions which means that they are not linearly independent.

Example 24, pg 152

$A = \begin{bmatrix} 3 & -6 & 9 & 0 \\ 2 & -4 & 7 & 2 \\ 3 & -6 & 6 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 5 & 4 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 0 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

For the Nul A
So basis for Nul A is

\[ v_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 1 \end{bmatrix} \]

For the basis for Col A we look for the pivot columns which are in bold

\[
A = \begin{bmatrix}
3 & -6 & 9 & 0 \\
2 & -4 & 7 & 2 \\
3 & -6 & 6 & -6
\end{bmatrix} \Rightarrow \begin{bmatrix}
1 & -2 & 0 & -6 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

and so we pick columns one and two.

\[ v_1 = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix} \]

Notice that the Col A is in \( \mathbb{R}^3 \) while the Nul A is in \( \mathbb{R}^4 \)